Speed of Convergence and Stopping Rules in an Iterative Planning Procedure for Nonconvex Economies

John P. Conley
Department of Economics
University of Illinois

BEBR

Speed of Convergence and Stopping Rules in an
Iterative Planning Procedure for Nonconvex Economies

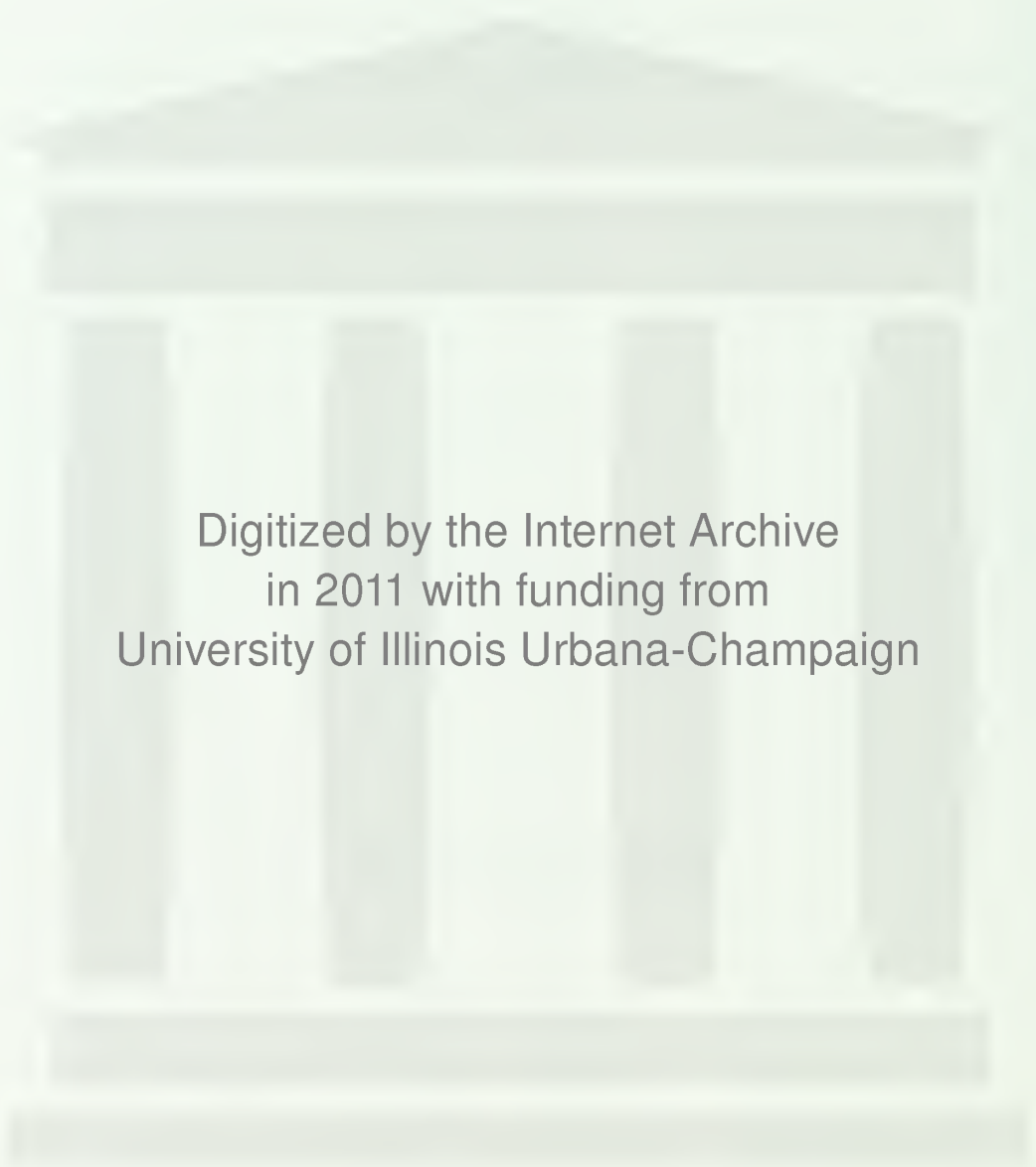John P. Conley

Department of Economics

Speed of Convergence and Stopping Rules in an Iterative Planning Procedure for Nonconvex Economies

John P. Conley†

Department of Economics
University of Illinois
Champaign, IL 61820

Revised: September 1992

Abstract

A new planning procedure is proposed for economies with free disposal in production, and monotonic social welfare functions. No convexity of any kind is required. The procedure takes the novel approach of searching for an optimum in the region of the unknown production set that the CPB knows the least about. This is different from the typical approach of searching in the region of the current estimate of the production set that is most preferred. This allows an upper bound to be calculated on the speed of convergence, and makes it possible to estimate the distance between the current tentative plan and the optimal plans. This is an essential feature if the procedure is likely to be stopped before it converges, and is absent in existing mechanisms. The procedure is also able to identify and discard information about regions of the production set that cannot contain optimal plans.

1. Introduction

With the changes taking place in the Eastern block, one might be led to wonder about the continued relevance of economic planning mechanisms. Most governments are turning, at least in part, to market based solutions to the problem of economic development. We know the price mechanism is well suited to finding efficient allocations in convex market economies. This begs the legitimate question of why a planning mechanism should be used at all in such economies. Indeed, some planning mechanisms, Weitzman (1970) for example, are highly suggestive of a market *tâtonement* process.

Unfortunately, convexity is an unreasonably strong restriction in many situations. In Public Finance, we know that externalities lead to fundamental nonconvexities in the production sets (see Starrett (1972)). In International Trade, there are concerns about increasing returns and learning by doing. In Industrial Organization, nonconvexities arise from natural monopoly and other forms of market power. Indivisibilities in large public projects can also lead to nonconvex production sets. In such cases, we have no assurance that the competitive equilibrium will even exist. Given this failure of the market, an alternative means must be employed to find socially optimal allocations.

This paper describes an iterative planning procedure which may be used in a broad class of economies. We require only that each firm in the economy has a closed production set and produces under conditions of free disposal, and the Central Planning Board (CPB) has a continuous and monotonic Social Welfare Function (SWF). We make no convexity assumptions of any kind.

Following the tradition of Malinvaud (1967), Weitzman (1970), and especially Crémer (1977), we frame the problem as a CPB trying to maximize a SWF over an unknown feasible set. To do this, the CPB asks a series of simple questions of the firms in the economy in order to generate successively better approximations of the

1

feasible set.

For the one firm, two good case, the procedure is roughly as follows (It may help to refer to figure 1.) The CPB starts off by choosing a point that overestimates and another point that underestimates a part of the Production Possibilities Frontier (PPF). Ideally this part of the PPF contains an optimal plan, but this is not required. We can imagine that the overestimate and the underestimate are the corners of a box around part of the PPF. Call the overestimate the "best point" of the box and the underestimate the "worst point" of the box.[1] We will show that the procedure converges to the most preferred point on the PPF in this initial box.[2]

Boxes have two edges radiating out of the best point (and of course two symmetric edges coming from the worst point.) This initial box is iteratively subdivided into a set of smaller boxes that surround the PPF in following way. Each iteration, the CPB finds the box whose longest edge is longer than every other box's longest edge (at the first iteration, there is only one box and so the choice is trivial.) We take this "biggest" box to be the region the CPB knows the least about. The CPB bisects the longest edge of the biggest box with a perpendicular plane and asks the firm to report a point in the intersection of the PPF and this plane. If such a point exists, the box is divided in half along its longest edge. Note that this also adds a new best point and worst point. If no such feasible point exists, it must be that the PPF either lies entirely above or entirely below the perpendicular plane within the box. In this case, the biggest box is divided in half and whichever of the two boxes does not contain any part of the PPF is discarded. Thus, each iteration, the box that is currently the largest is divided in half along its largest edge.

---

[1] We call the overestimate the best point since by monotonicity it is the most preferred point in the box. The reverse is true for the worst point.

[2] Thus, by choosing a larger initial box, the CPB makes it likely that the procedure will converge to a better plan. However, this also slows down convergence, as a larger part of the PPF has to be considered.

The procedure uses the information acquired above as follows. The points on the PPF that the firm reports allow the CPB to find an underestimate (called "lower estimate" in this paper) of the production set. By free disposal, the CPB knows that everything that lies below a point that is feasible is also feasible. Thus, the *comprehensive hull* [3] of all of the known feasible points may be taken as the lower estimate. The set of best points described previously is constructed such that taking their comprehensive hull gives a set that contains the PPF and thus is an overestimate (called "upper estimate" in this paper) of the production set. The most preferred point on the lower estimate at any given iteration is a feasible point which is taken as the current tentative plan. Note that since this lower estimate expands each iteration, the set of tentative plans is monotonically increasing in the preference ordering. The procedure has one last feature which reduces its informational requirements. If there is a best point that is inferior under that SWF to a tentative plan, than that best point clearly cannot lie above an optimal plan. Thus the CPB may discard boxes dominated by such inferior best points.[4]

In summery, the algorithm proceeds as follows. Each iteration, the CPB finds the most preferred point on the current lower estimate and names this the tentative plan. It then discards all boxes whose best point is inferior to the tentative plan. Next, the CPB finds the "biggest" box remaining and divides it in half along its longest edge. If both of the two new boxes contain parts of PPF, then both are retained, and the firm reports any point in the intersection of the PPF and the plane that bisects and is perpendicular to the longest edge. Appropriate new points are added to the set of best, worst and feasible points. Otherwise, the box that does not contain a part of the PPF is discarded and the set of best and worst points is updated appropriately.

---

[3] See the next section for a precise definition of comprehensive hull.

[4] This identification of irrelevant information is a novel feature of this mechanism, as far as we know.

Most planning procedures ask firms to refine the CPB's information in the region of the current estimate of the production set that is most preferred under the SWF. However, in a nonconvex environment, there is no particular reason to believe that the optimal plan is close to the currently most preferred region. Indeed, the indifference curve through the optimal plans may come very close to many, widely dispersed areas of the PPF if neither is convex. In the mechanism described in this paper, the CPB asks each firm to refine the board's information in the region it knows the least about. This is a novel approach, as far as we know, and yields several advantages.

In general, most mechanisms cannot be expected to converge to an optimal plan in a finite number of iterations. Even if this were not the case, Bennett (1985) points out that time shortages may require that the procedure be stopped before convergence, and the current tentative plan taken as the final approximation of an optimal plan. Given this real world constraint, it is essential that CPB know how quickly the various planning algorithms start suggesting tentative plans that are satisfactorily close to an optimal plan. Without this information, there is no reasonable way to choose which algorithm to use.

This problem has led several authors to investigate the question of speed of convergence. One approach is to show that one procedure is always faster than another. For example, Clark (1989) compares the procedures of Malinvaud (1967) and Chander (1978). He shows that Malinvaud's is at least as fast as Chander's in a certain class of economies, although Malinvaud's procedure has a higher informational requirement. Unfortunately, it is very difficult to generalize this type of comparison to other procedures. It would be quite surprising if it were possible to rank all the known procedures on the basis of speed, given the diversity of approaches taken in this literature. In any event, such theorems still leave the CPB ignorant about the actual speed of convergence of any given mechanism.

An alternative approach is to conduct computer simulations in order to test

4

which procedure is faster. Clark gives some very interesting results of such an experiment with the Malinvaud and Chander procedures. However, as Chander and Kundu (1991) point out, the results are highly dependent on the class of examples considered. To be able to make useful assertions like "procedure α converges faster than procedure β, x percent of the time," we would need to have a measure over the class of reasonable economies. Defining this class, much less a sensible measure, seems an impossible task, especially if we do not restrict attention to convex economies. So while these results are suggestive, and interesting, it would be overly hasty to choose a procedure based on simulations alone.

We take a different, and we argue, much more complete approach in this paper. It is easy to find an upper bound on the longest edge of the biggest box at any given iteration. Since this tells us the maximal size of the boxes that surround the PPF, we can use this to find an upper bound on the speed of convergence in several different ways. This bound may then be employed to make direct comparisons with other procedures for which a bound on speed of convergence is known. In addition, the actual degree of ignorance (size of the largest box) may be calculated exactly as the algorithm progresses. This makes it possible to set up informed stopping rules of the form: stop searching for an optimum if the current tentative plan is within a given distance of an optimal plan under some measure (eg. within a certain utility value, or within a certain ϵ-ball in the goods space).

Thus, the major advantage of this mechanism is that the novel method of information acquisition makes it possible to directly estimate the speed of convergence, and to set up sensible stopping rules. It is not clear how one would do this for existing procedures defined for nonconvex economies such as Henry and Zylberberg (1978) or Crémer (1977, and 1983).

2. Definitions and assumptions

This section lays out the basic definitions and assumptions that are used in what follows. We consider a CPB with a complete and transitive social preference ordering, $\succeq$, over a closed consumption set with m goods: $X \subseteq R^m$. The CPB is endowed with $\omega \in \Re^m$ at the outset.[5] We assume:

A1) The preference relation $\succeq$ is continuous.

A2) If $x > x'$ then $x \succ x'$.[6]

   (Preferences are strongly monotonic.)

The weak upper and lower contour sets of this relation for any subset $Z \subseteq R^m$ are denoted by $U(Z)$ and $L(Z)$ respectively. Formally:

$$U(Z) \equiv \{x \in X \mid x \succeq x' \text{ for some } x' \in Z\}$$

and,

$$L(Z) \equiv \{x \in X \mid x \preceq x' \text{ for some } x' \in Z\}.$$

The boundary of a set is denoted by "∂". Under these two assumptions, $\partial L(Z)$ is the indifference surface through the infimum of Z under the preference ordering. These assumptions also imply that if $x \in X$, and $x' \geq x$ then $x \in X$.

There are k firms in the economy with production sets denoted by Y^ℓ for $\ell = 1, \ldots, k$. We make two assumptions on firms' production sets:

B1) If $y \in Y^\ell$, then $y' \leq y$ implies $y' \in Y^\ell$ for $\ell = 1, \ldots, k$.

   (Free disposal in production, or identically, comprehensiveness of Y^ℓ.)

---

[5] It is possible to reformulate the problem such that the firms own the initial endowments without changing the procedure in any substantial way. However, this is a planning problem, so there does not seem to be any gain from doing so.

[6] The three types of vector inequalities are $\gg, >, and \geq$.

B2) Y^ℓ is closed.

We dispense with all convexity assumptions, and depend on monotonicity and free disposal instead. Without convexity, closedness of each firms' feasible set is not enough to conclude that the set of feasible allocations for the economy is also closed in general. In this algorithm, however, the search for an optimal production plan is limited to a compact subset of the each firms' production set. Since the sum of these compact subsets of Y^ℓ is the only part of the global production set that is considered, the relevant part of the global production set is compact.

The usual notion of set summation is used:

$$\sum_{i=1}^{t} Z^i \equiv \{x \in \Re^m \mid z = z^1 + \cdots + z^t \text{ and } z^i \in Z^i \text{ for } i = 1, \ldots, t\}.$$

Each global object, z, (a global feasible production plan, for example) is the sum of one element from each of the Z^ℓ's. That is, $z = z^1 + \ldots + z^k$ where $z^\ell \in Z^\ell$ for $\ell = 1, \ldots, k$. We denote by $\bar{z}$ this set of k elements that sum to z. For example, if we use this notation to indicate the decomposition of a set of vectors, Z, this becomes:

$$\bar{Z} \equiv \left\{ \begin{array}{c} \bar{z}_1 \\ \vdots \\ \bar{z}_i \\ \vdots \\ \bar{z}_s \end{array} \right\} \equiv \left\{ \begin{array}{ccccc} z_1^1 & z_1^2 & \cdots & z_1^{k-1} & z_1^k \\ \vdots & \vdots & & \vdots & \vdots \\ z_i^1 & z_i^2 & \cdots & z_i^{k-1} & z_i^k \\ \vdots & \vdots & & \vdots & \vdots \\ z_s^1 & z_s^2 & \cdots & z_s^{k-1} & z_s^k \end{array} \right\} \in \Re^{m \times k \times s},$$

where $z_j^\ell \in Z^\ell$ for all j and all $\ell = 1, \ldots, k$, and $\sum_{\ell=1}^{\ell} z_j^\ell = z_j$.

Since we will not consider the case of production externalities,[7] the set of feasible plans for the economy is:

$$Y \equiv \left\{ y \in \Re^m \mid y = \sum_{\ell=1}^{k} Y^k + \omega \right\}.$$

---

[7] This generalization is easily accomplished without any change in the model except the definition of the global feasible set. However, the notational cost (firm i's observation of firm j's output of good k) would be heavy in an already notationally dense paper.

This simply says that a plan is feasible for the economy if it is the sum of the endowment and a global production plan which can be decentralized in such a way that each firm is asked to produce a feasible net-output vector. We will use the convention that superscripts refer to the firms, and omission of superscripts indicate a global object. Thus $y^\ell \in Y^\ell$ is a production plan for firm ℓ, while $y \in Y$ is a production plan for the economy.

The procedure depends heavily on the notions of taking comprehensive hulls of various sets. For any set $Z \subseteq \Re^m$, define the *comprehensive hull of Z* as follows:

$$ch(Z) \equiv \{x \in \Re^m \mid \exists\, z \in Z \text{ s.t. } x \leq z\}$$

This is the set of vectors in $\Re^m$ that are weakly dominated by some vector in Z. We will also have need of the notion of an *inverse comprehensive hull of Z* in explaining the mechanism:

$$ich(Z) \equiv \{x \in \Re^n \mid \not\exists\, z \in Z \text{ s.t. } x \gg z\}$$

This is the set of vectors in $\Re^m$ that do not strictly dominate some vector in Z. Note that both the comprehensive hull and inverse comprehensive hull of a finite collection of vectors is closed.

Finally, consider the following notion of minimal "distance":[8]

$$\nu(A, B) \equiv \inf_{y \in B} \inf_{x \in A} \| x - y \|$$

where A and B are sets in $\Re^m$, and $\| \bullet \|$ denotes the Euclidean norm. ν gives the infimum of length of the smallest gap between two sets. Thus, if A and B intersect, then $\nu(A, B) = 0$. A useful property of ν is given in Lemma 1.

Lemma 1. *If $A' \subseteq A \subset \Re^m$ and $B' \subseteq B \subseteq \Re^m$, are all bounded then, $\nu(A', B') \geq \nu(A, B)$.*

Proof/

---

[8] Note that ν is not a measure of distance in a mathematical sense since it does not satisfy the triangle inequality.

By the definition of ν, and of compact sets, there must be two points, $a' \in$ closure(A'), and $b' \in$ closure(B') such that $\nu(A', B') \geq \nu(a', b')$. But then, $a' \in$ closure(A) and $b' \in$ closure(B). So $\nu(A', B') \geq \nu(a', b') \geq \nu(A, B)$.

●

3. The Algorithm

The algorithm is based on the observation that it is possible to construct a lower bound as well as an upper bound on the production set of each firm using the information acquired through the Crémer procedure. These estimates can then be used jointly to narrow the area of search with each iteration. Basically, this is done by removing from consideration all parts of the overestimate of the production set that are inferior to some part of the underestimate. In the Crémer procedure, and in most other iterative planning procedures, the entire feasible set must be considered as potentially containing an optimum unless the procedure actually converges.

Crémer constructs an overestimate of the production set of each firm by using free disposal to conclude that points that strictly dominate elements of the PPF cannot themselves be feasible. Formally, he constructs an *Upper Bound on the Production Set of the ℓth firm at iteration n* thus:

$$\text{UBPF}\,_n^\ell \equiv ich(X_n^\ell) \bigcap ch(b_0^\ell)$$

where X_n^ℓ is the set of points known to be on the ℓth firm's being PPF at the nth iteration, and b_0^ℓ is a vector that dominates at least one optimal production plan. Note that $UBPF_n^\ell$ is a closed and comprehensive set as the intersection of two such sets.

9
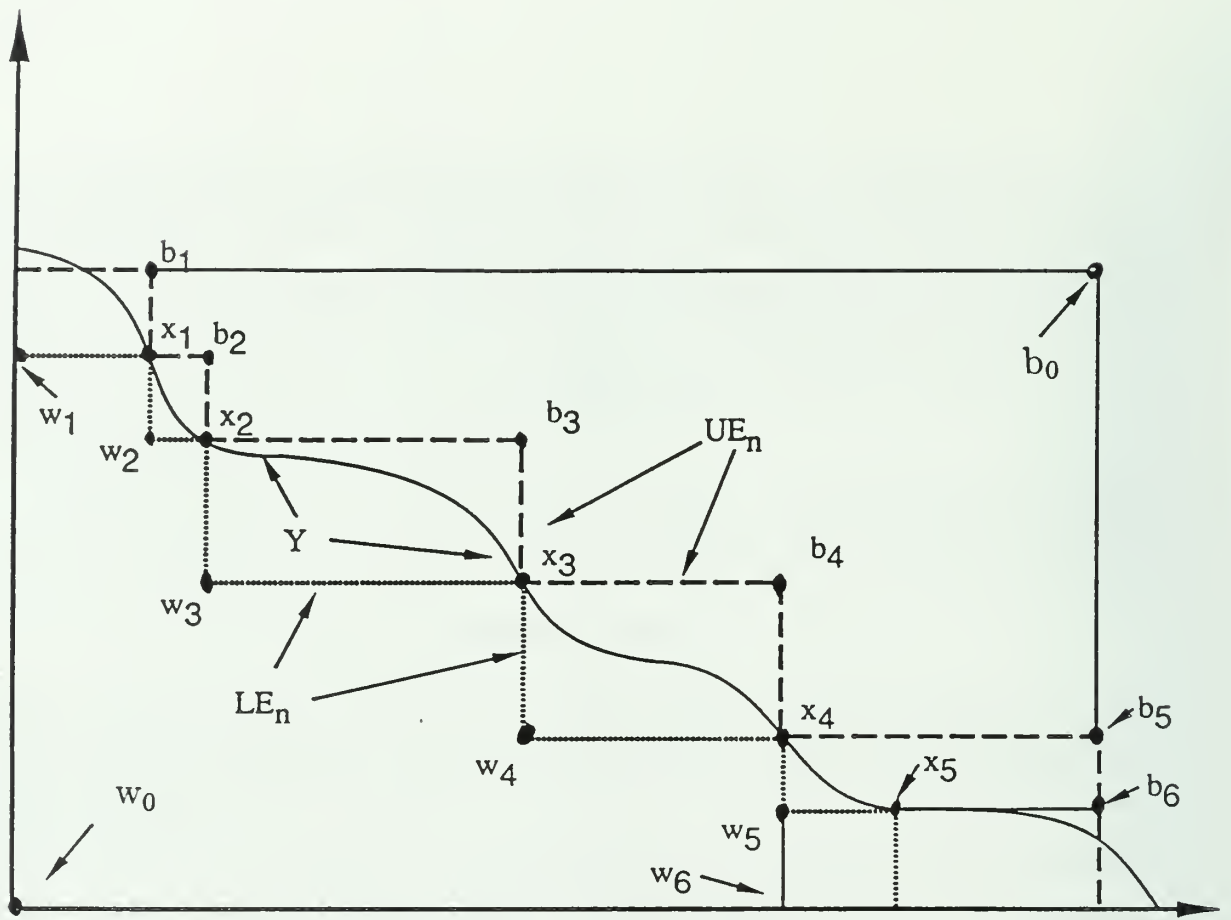
$$X_n^\ell = \{x_1^\ell, x_2^\ell, x_3^\ell, x_4^\ell, x_5^\ell\}$$

$$B_n^\ell \doteq \{b_1^\ell, b_2^\ell, b_3^\ell, b_4^\ell, b_5^\ell, b_6^\ell\}$$

$$W_n^\ell = \{w_1^\ell, w_2^\ell, w_3^\ell, w_4^\ell, w_5^\ell, w_6^\ell\}$$

$$C_n^\ell = \{(b_1^\ell, w_1^\ell); (b_2^\ell, w_2^\ell); (b_3^\ell, w_3^\ell); (b_4^\ell, w_4^\ell); (b_5^\ell, w_5^\ell); (b_6^\ell, w_6^\ell)\}$$

Figure 1

Free disposal can obviously be applied in the other direction as well. If a point, x, is known to be feasible, then all points that x dominates must also be feasible. The CPB may therefore take the comprehensive hull of all known feasible points as a lower bound on the production set. Thus, we define the *Lower Estimate of the ℓth firm's production set at iteration n* as:

$$LE_n^\ell \equiv ch(X_n^\ell).$$

Figure 1 illustrates these upper and lower bounds on a firm's production set.

10

Notice that the set theoretical difference between them is a union of "cubes".[9] This will also be true in higher dimension. Each cube is uniquely characterized by a point b, which dominates all other points in the cube (called the *best* point), and a point w, which is dominated by all other points in the cube (called the *worst* point) via the correspondence:

$$\mathcal{C}(b, w) \equiv \{x \in \Re^m \mid b \geq x \geq w\}.$$

Observe that Crémer's upper bound on the production set can also be found by taking the comprehensive hull of all the best points. Since the set of best points is explicitly constructed in the algorithm described in this paper, it is natural that the upper bound on the production set, called the *Upper Estimate* here, should be defined accordingly:

$$UE_n \equiv ch(B_n^\ell).$$

The algorithm itself involves the iterative updating of four sets of quantity vectors and the naming of a tentative production plan, p_n^ℓ, for each firm, for each iteration. The first of these is C_n^ℓ, a set of ordered pairs $\{(b_1^\ell, w_1^\ell), \ldots, (b_i^\ell, w_i^\ell), \ldots, \} \equiv \{c_1^\ell, \ldots, c_i^\ell, \ldots, \}$ such that when these pairs are used to form cubes, their union equals the set theoretical difference between UE_n^ℓ and LE_n^ℓ. The next two are B_n^ℓ and W_n^ℓ, the sets of best and worst points in C_n^ℓ (that is, the set of all first and second elements, respectively, in the set of ordered pairs C_n^ℓ). It is notationally convenient to construct these sets separately even though all the information contained in them is also contained in C_n^ℓ. Last is X_n^ℓ, which is a set of points known to lie exactly on firms' PPF's.

All of the sets above are constructed individually for each firm. These are aggregated to form overestimates and underestimates of the global PPF as follows:

---

[9] The use of the word "cube" is not meant to imply that these objects have equal sides, or that they are three dimensional. We use "cube" as a substitute for the more accurate, but awkward "hyper-rectangle".

$$B_n \equiv \sum_{\ell=1}^{k} B_n^{\ell}$$

$$UE_n \equiv ch(B_n)$$

$$X_n \equiv \sum_{\ell=1}^{k} X_n^{\ell}$$

$$LE_n \equiv ch(X_n).$$

Thus, the global upper and lower estimates are of the set of feasible global net output vectors, not of the production set itself. The set of global feasible allocations is found by adding the ω to the upper and lower estimates. We follow this convention for notational convenience.

The CPB must make an initial over and underestimate of the production set for each of the firms. These initial estimates may be as broad or narrow as desired. We show that the algorithm converges to the best plan possible within these limits. In particular, if we assume that the CPB knows enough to be able to over and underestimate a part of the production set that contains an optimal plan, then the algorithm converges to the true optimum. Crémer makes an assumption in this spirit. Our approach is make explicit the trade off between speed of convergence and the possibility of finding a better plan that choosing the boundaries of search entails.

Formally, for each firm, the CPB chooses initial over and underestimates of the production sets, (b_0^{ℓ}, w_0^{ℓ}), such that:

$$\forall \, \ell = 1, \ldots, k, \ b_0^{\ell} \notin Y^{\ell}, w_0^{\ell} \in Y^{\ell}, \text{ and } b_0^{\ell} >> w_0^{\ell}.$$

The algorithm is then initialized as follows:

$$C_0^{\ell} \equiv \{c_0^{\ell}\} \equiv \{(b_0^{\ell}, w_0^{\ell})\} \text{ for } \ell = 1, \ldots, k,$$

12

$$B_0 \equiv \left\{ \sum_{\ell=1}^{k} b_0^\ell \right\},$$

$$X_0^\ell \equiv \{w_0^\ell\} \text{ for } \ell = 1, \ldots, k \qquad X_0 \equiv \{w_0\} \equiv \left\{ \sum_{\ell=1}^{k} w_0^\ell \right\}$$

$$p_0^\ell \equiv \{w_0^\ell\} \text{ for } \ell = 1, \ldots, k \qquad p_0 \equiv \{w_0\} \equiv \left\{ \sum_{\ell=1}^{k} w_0^\ell \right\},$$

where p_0^ℓ is the tentative production plan for firm ℓ at iteration 0, and p_0 is the corresponding global tentative plan.

Neither the set of global cubes nor the set of global worst points is collected. This is because the global best points are constructed in such a way that their comprehensive hull gives an overestimate of the production set. The comprehensive hull of the known points X_n, on the other hand, gives an underestimate. These two estimates are all that we need at the global level. Cubes are useful only in that their size may be used as a measure of the CPB's ignorance about a particular firm's production possibilities. The CPB will end up asking the firms questions about the part of the PPF contained in the "biggest" cube. Since no analogous question is ever asked at the global level, global cubes, and consequently, global worst points are not needed.

We will show that the algorithm converges to the best plan within the search boundaries defined by above. Formally, let $Y : \Re^{2mk} \rightarrow \Re^m$ be the correspondence that gives the subset of feasible global net output vectors examined when the CPB chooses c_0, as his search area:

$$Y(\bar{c}_0) \equiv Y(c_0^1, \ldots, c_0^k) \equiv \left\{ y = \sum_{\ell=1}^{k} y^\ell, \middle| \forall \ell = 1, \ldots, k, y^\ell \in \left\{ \mathcal{C}(b_0^\ell, w_0^\ell) \bigcap Y^\ell \right\} \right\}.$$

Then let $X^* : \Re^{2mk} \times \rightarrow \Re^m$ be the correspondence that gives the set of optimal, global net output vectors in $Y(\bar{c}_0)$:

$$X^*(\bar{c}_0) \equiv \{ y \in Y(\bar{c}_0) \mid y + \omega \succeq y' + \omega \ \forall \ y' \in Y(\bar{c}_0) \}.$$

13

Again, these correspondences give feasible and optimal net output vectors, not allocations. We are finally ready to define the algorithm. Each iteration is broken down into five steps. Figure 2 gives an illustration for the one firm case.

Step 1) The first step for any given iteration, n, is to take the most preferred point on the global lower estimate for the previous iteration, LE_{n-1}, as the global tentative plan p_n. Since $LE_{n-1} \equiv ch(X_{n-1})$, and the CPB's preferences are monotonic, this maximization must take place at some element of X_{n-1}. Each element $x \in X_{n-1}$ has a known decomposition $\bar{x} = (x^1, \ldots, x^k)$. Thus, p_n may be decomposed into $\bar{p}_n$, which in turn is taken as a specification of a tentative production plan for each firm. Formally:

1) Some $p_n \in \{x \in LE_{n-1} \mid x + \omega \succeq y + \omega \ \forall \ y \in LE_{n-1}\}$ is chosen and $\bar{p}_n$ is declared to be the tentative plan for the economy.

Step 2) The second step is to find all elements of B_{n-1} that are strictly inferior to p_n, and discard them to form the new set $\tilde{B}_n$. We do this because any element of the overestimate that is inferior to a feasible point p_n certainly cannot lie above an optimal plan. Such points may therefore be safely removed from future consideration. Likewise, elements that are known to be feasible, but which are nevertheless dominated by a discarded element of B_n, may be thrown away. Formally:

2) $\tilde{B}_n \equiv \{b \in B_{n-1} \mid p_n + \omega \preceq b + \omega\}$
   $\tilde{X}_n \equiv \left\{x \in X_{n-1} \mid b \geq x \text{ for some } b \in \tilde{B}_n\right\}$

Step 3) Having found and discarded the irrelevant elements of the global sets B_{n-1}, and X_{n-1}, the next step is to find the irrelevant elements of the sets collected from each of the firms. Suppose that a best point for a firm, $b^\ell \in B^\ell_{n-1}$, is *only* used to create global best points which are known to be inferior to a feasible point. Then clearly, the firm will never be called upon to produce b^ℓ, or any point that
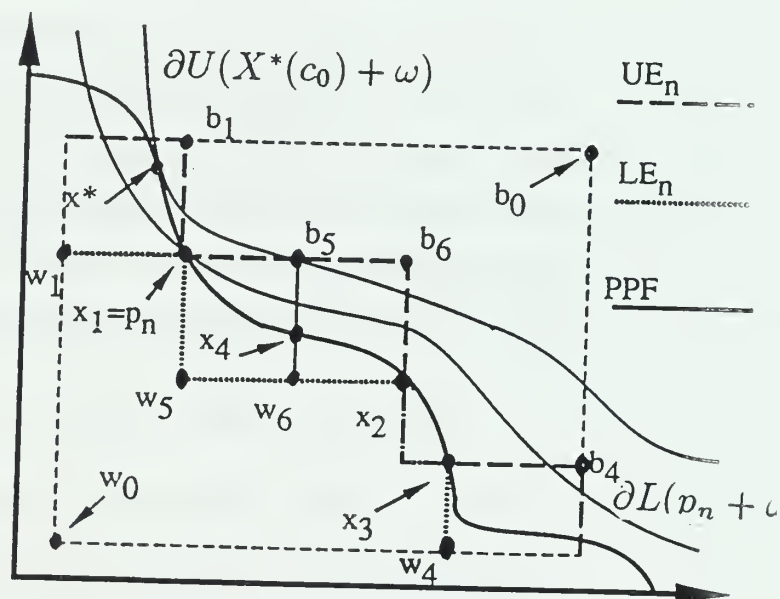
14
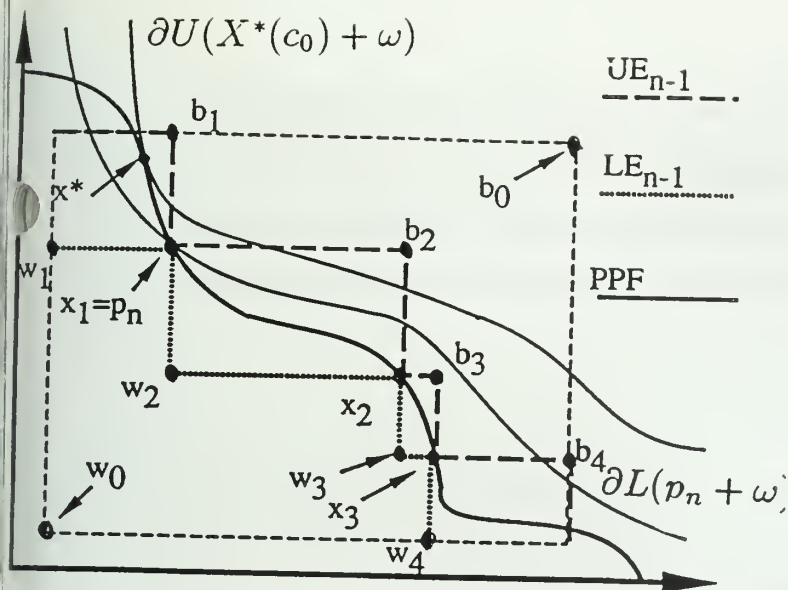
2a) Step 1: x_1 is the most prefered point on LE_n and is declared the tenative plan, p_n.
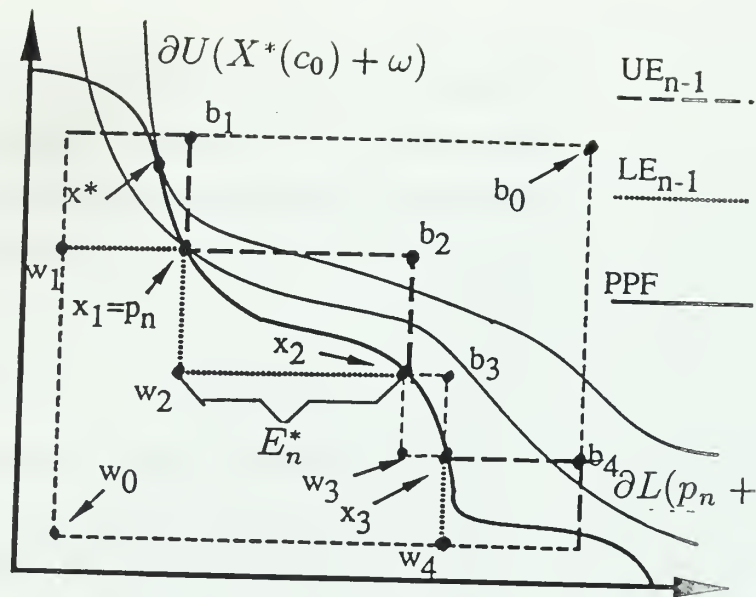
2b) Steps 2 and 3: b_3 is found to be inferior to p_n and so b_3, w_3, and c_3 are discarded.
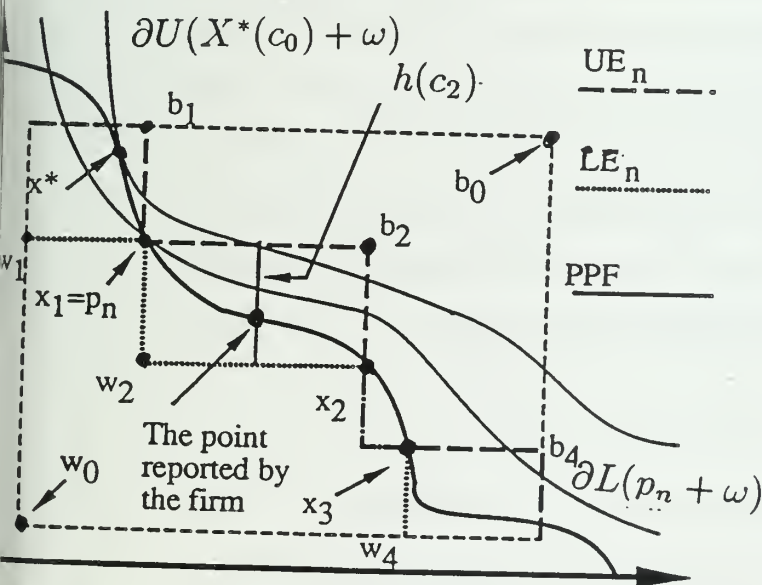
2c) Step 4: The xth edge of the second cube is found to be the longest. Case 1 holds, and the firm reports a point in the in the intersection of $h(c_2)$ and the PPF

2d) Step 5: c_2 is divided in half along the x-axis and replaced with the half new cubes, and appropriate best and worst points added to B_n and W_n.

Figure 2: The new procedure

15

b^ℓ dominates. This is identical to saying that if b^ℓ is only used to create global points that are not in the set $\tilde{B}_n$, then it may be discarded. Furthermore, all points of the sets W_{n-1}^ℓ and C_{n-1}^ℓ that are associated with discarded best points, and all elements of X_{n-1}^ℓ that are dominated by discarded points of B_{n-1}^ℓ may be discarded. Formally:

3) $(\tilde{B}_n^1, \ldots, \tilde{B}_n^k) \equiv (\tilde{\bar{B}}_n),$[10]

    and for each $\ell = 1, \ldots, k$:

$$\tilde{W}_n^\ell \equiv \{w^\ell \in W_{n-1}^\ell \mid \exists\, (b^\ell, w^\ell) \equiv c^\ell \in C_{n-1}^\ell \text{ where } b^\ell \in \tilde{B}_n^\ell\}$$

$$\tilde{C}_n^\ell \equiv \{(b^\ell, w^\ell) \equiv c^\ell \in C_{n-1}^\ell \mid b^\ell \in \tilde{B}_n^\ell\}$$

$$\tilde{X}_n^\ell \equiv \{x^\ell \in X_{n-1}^\ell \mid b^\ell \geq x^\ell \text{ for some } b^\ell \in \tilde{B}_n^\ell\}.$$

Step 4) Having discarded irrelevant information about each of the firms' production sets, the next step is to gain more information about the parts that remain. In this algorithm, the CPB asks for information about the region he knows the least about. Notice that each cube has m edges radiating out from its best point. The CPB is interested in the cube whose longest edge is longer than every other cube's longest edge. Let the function $ILE : \Re^m \times \Re^m \to \{1, \ldots, m\}$ give the Index of the Longest Edge of a cube where ties are broken by taking the lowest index:

$$ILE(c^\ell) \equiv \Big\{ j \in (1, \ldots, m) \mid \forall\, i = 1, \ldots, m \ \ b^{j,\ell} - w^{j,\ell} \geq b^{i,\ell} - w^{i,\ell},$$

$$\text{and } j < i \ \forall\, i, j = 1, \ldots, m \text{ s.t. } b^{j,\ell} - w^{j,\ell} = b^{i,\ell} - w^{i,\ell} \Big\}.$$

In all cases the first superscript refers to a component of a vector while the second refers to the firm. Let $E_n^{\ell *}$ denote the length of the longest edge any cube in $\tilde{C}_n^\ell$:

$$E_n^{\ell *} \equiv \Big\{ b^j - w^j \mid b^j - w^j \geq \hat{b}^i - \hat{w}^i \ \forall\, c, \hat{c} \in \tilde{C}_n^\ell \text{ and } \forall\, i, j = 1, \ldots, m \Big\}.$$

---

[10] Any given element b^ℓ may be summed with many different permutations of best points from other firms to form the set of global best points. Thus, when the set of global best points is decomposed, there may be a great deal of repetition in the list of best points for each firm.

Note that if $E_n^{\ell *}$ is the length of the longest edge of any cube in $\tilde{C}_n^\ell$, it is also an upper bound on the length of the longest edge of any cube in C_n^ℓ, since no cube grows in size as $\tilde{C}_n^\ell$ is converted into C_n^ℓ. Correspondingly, E_n^* is an upper bound on the length of the longest edge of any global cube:

$$E_n^* \equiv \sum_{\ell=1}^{k} E_n^{\ell *}.$$

Let $\tilde{C}_n^{\ell *}$ be the set of cubes in $\tilde{C}_n^\ell$ whose longest edge is at least as long $E^{\ell *}$:

$$\tilde{C}_n^{\ell *} \equiv \left\{ c \in \tilde{C}_n^\ell \mid b^j - w^j = E^{\ell *} \text{ and } j = ILE(c) \right\}.$$

For any cube c, consider the hyperplane which bisects the cube, and is perpendicular to the longest edge. We denote the correspondence that gives this hyperplane by $h : \Re^{2m} \to \Re^m$:

$$h(c^\ell) \equiv \left\{ x \in \Re^m \mid \text{ for } j = ILE(c^\ell), x^j = \frac{(b^{j,\ell} + w^{j,\ell})}{2} \right\}.$$

In the algorithm, for each firm, an arbitrary element $c^\ell \in C_n^{\ell *}$, is chosen by the CPB, and the firm is asked to report some point on the PPF within the cube c^ℓ, and on $h(c^\ell)$. That is to say, the firm is asked to report some point in the set:

$$PPF(c^\ell) \equiv \left\{ h(c^\ell) \cap \partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell) \right\}.$$

If such a point exists, it is added to the set $\tilde{X}_n^\ell$ to create X_n^ℓ. The set, $PPF(c^\ell)$, however, may be empty. In this case, the PPF must be either completely above or completely below the hyperplane within the cube. This is because of the assumption that Y^ℓ is closed, which implies that the PPF is continuous. If either one of the above is true, the firm reports this instead of a point. In both cases, X_n^ℓ is set equal to $\tilde{X}_n^\ell$. It is now possible to conclude that the procedure is well defined since this is the only question that the firms are ever asked, and cases (a), (b), and (c), below, are exhaustive. Formally:

17

4) For each $\ell = 1, \ldots, k$, $c^\ell \in \tilde{C}_n^{\ell*}$ is chosen, and the firm is asked to report any $x^\ell \in PPF(c^\ell)$. The firm makes one of three responses:

   (a) $PPF(c^\ell) \neq \emptyset$, and the firm reports some element of the set, x^ℓ.

   (b) $PPF(c^\ell) = \emptyset$ and $x < y \ \forall \ x \in C(b^\ell, w^\ell) \cap h(c^\ell)$ and $y \in \partial Y^\ell \cap C(b^\ell, w^\ell)$.

   (c) $PPF(c^\ell) = \emptyset$ and $x > y \ \forall \ x \in C(b^\ell, w^\ell) \cap h(c^\ell)$ and $y \in \partial Y^\ell \cap C(b^\ell, w^\ell)$.

   In case (a), $X_n^\ell \equiv \left\{ \tilde{X}_n^\ell \cup x^\ell \right\}$.

   In case (b) and (c), $X_n^\ell \equiv \tilde{X}_n^\ell$.

Step 5) The fifth and final step of the algorithm, which is illustrated in figure 3, is to update the sets B_{n-1}^ℓ, W_{n-1}^ℓ, and C_{n-1}^ℓ. This was partially accomplished in step 3 when these sets were turned into $\tilde{B}_n^\ell$, etc. The updating is completed in different ways depending on what the firm reports at step 4. The first possibility is that the set $PPF(c^\ell)$ is not empty, and the firm reports some element x^ℓ to the CPB. In this case, the cube c^ℓ is divided in half along the j axis, and c^ℓ is replaced with the two new cubes. Appropriate best and worst points are added to create B_n^ℓ and W_n^ℓ. The second possibility is that $PPF(c^\ell)$ is empty and $h(c^\ell) \cap C(b^\ell, w^\ell)$ is below $\partial Y^\ell \cap C(b^\ell, w^\ell)$. In this case c^ℓ is truncated by moving its worst point halfway up the longest edge of the cube. The last possibility is that $PPF(c^\ell)$ is empty and $h(c^\ell) \cap C(b^\ell, w^\ell)$ is above $\partial Y^\ell \cap C(b^\ell, w^\ell)$. Here, the opposite is done. The best point of the cube is moved halfway down the longest edge. Formally:

5) For each $\ell = 1, \ldots, k$, if case (a) obtains:

   $B_n^\ell \equiv \{ (\tilde{B}_n^\ell \setminus b^\ell) \cup b_\alpha^\ell \cup b_\beta^\ell \}$

   where $b_\alpha^\ell \equiv b^\ell$,
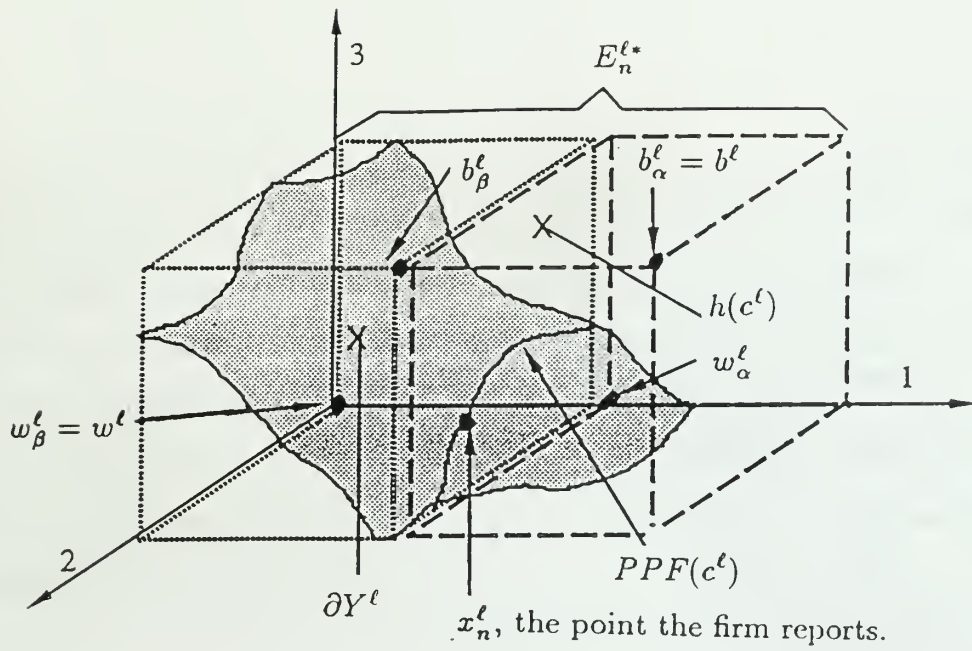
   and $b_\beta^\ell \equiv (b^{1,\ell}, \ldots, b^{j-1,\ell}, \frac{b^{j,\ell} + w^{j,\ell}}{2}, b^{j+1,\ell}, \ldots, b^{m,\ell})$;

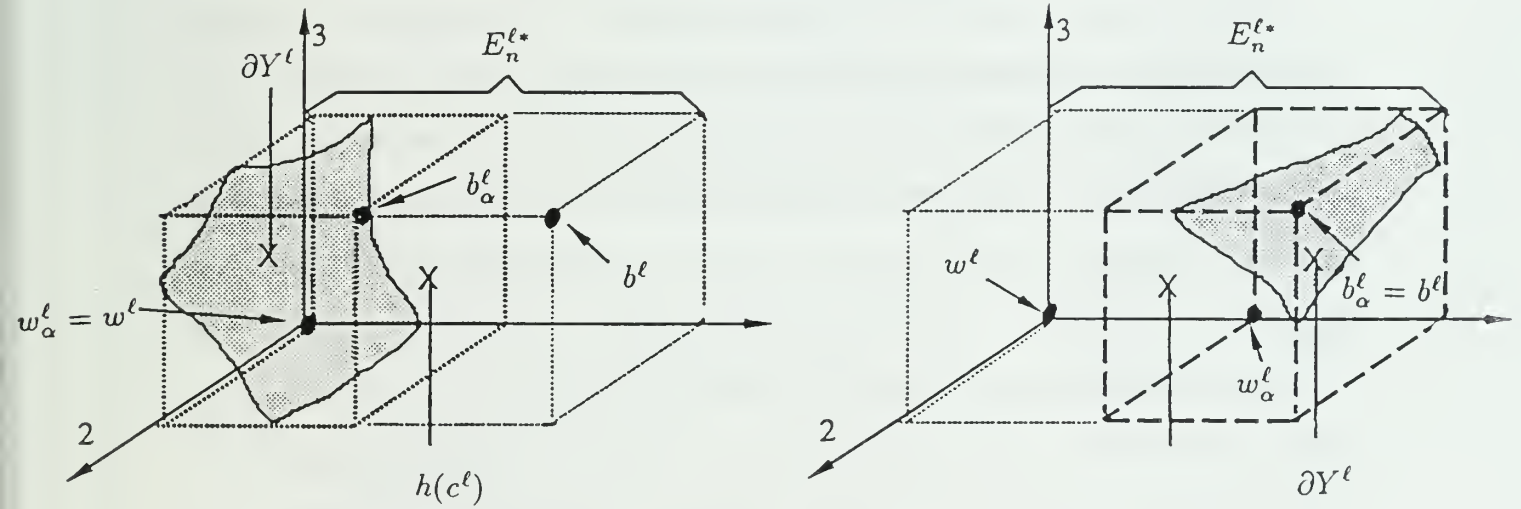 $W_n^\ell \equiv \{ (\tilde{W}_n^\ell \setminus w^\ell) \cup w_\alpha^\ell \cup w_\beta^\ell \}$

   where $w_\alpha^\ell \equiv (w^{1,\ell}, w^{2,\ell}, \ldots w^{j-1,\ell}, \frac{b^{j,\ell} + w^{j,\ell}}{2}, w^{j+1,\ell}, \ldots, w^{m,\ell})$,

   and $w_\beta^\ell \equiv w^\ell$;

   $C_n^\ell \equiv \{ (\tilde{C}_n^\ell \setminus c^\ell) \cup c_\alpha^\ell \cup c_\beta^\ell \}$.

18

3a) In this example, assume that $E_n^{\ell*} = (b^{1,\ell} - w^{1,\ell})$. Then the cube c^ℓ is to be divided according to case (a) since $PPF(c^\ell) \neq \emptyset$. As can be seen in the figure, c is divided exactly in half along the first edge creating two new cubes c_α^ℓ and c_β^ℓ, whose union equals c^ℓ.

3b) Here. $PPF(c^\ell) = \emptyset$ and $\partial Y^\ell \cap c^\ell$ is below $h(c^\ell)$. Case (b) obtains, and w^ℓ is moved up to w_α^ℓ and the dotted area is discarded

3c) Here, $PPF(c^\ell) = \emptyset$ and $\partial Y^\ell \cap c^\ell$ is above $h(c^\ell)$. Case (c) obtains, and b^ℓ is moved down to b_α^ℓ and the dotted area is discarded

Figure 3: The three ways to divide a cube

19

In case (b):

$$B_n^\ell \equiv \{(\tilde{B}_n^\ell \setminus b^\ell) \cup b_\alpha^\ell\},$$

where $b_\alpha^\ell \equiv b^\ell$;

$$W_n^\ell \equiv \{(\tilde{W}_n^\ell \setminus w^\ell) \cup w_\alpha^\ell\}$$

where $w_\alpha^\ell \equiv (w^{1,\ell}, \ldots, w^{j-1,\ell}, \frac{b^{j,\ell} + w^{j,\ell}}{2}, w^{j+1,\ell}, \ldots, w^{m,\ell})$;

$$C_n^\ell \equiv \{(\tilde{C}_n^\ell \setminus c^\ell) \cup c_\alpha^\ell\}.$$

In case (c):

$$B_n^\ell \equiv \{(\tilde{B}_n^\ell \setminus b^\ell) \cup b_\alpha^\ell\}$$

where $b_\alpha^\ell \equiv (b^{1,\ell}, \ldots, b^{j-1,\ell}, \frac{b^{j,\ell} + w^{j,\ell}}{2}, b^{j+1,\ell}, \ldots, b^{m,\ell})$;

$$W_n^\ell \equiv \{(\tilde{W}_n^\ell \setminus w^\ell) \cup w_\alpha^\ell\}$$

where $w_\alpha^\ell \equiv w^\ell$;

$$C_n^\ell \equiv \{(\tilde{C}_n^\ell \setminus c^\ell) \cup c_\alpha^\ell\}.$$

These five steps constitute one complete iteration of the algorithm. The next few lemmata substantiate the claim that the comprehensive hulls of B_n and X_n do indeed give upper and lower estimates of the production set. Lemma 2 says that no part of the PPF of a firm that is in some cube after step 3 is removed in step 5.

Lemma 2. For all $y^\ell \in \partial Y^\ell$, if $y \in \bigcup_{c \in \tilde{C}_n^\ell} \mathcal{C}(b, w)$, then $y \in \bigcup_{c \in C_n^\ell} \mathcal{C}(b, w)$.

Proof/

It must be shown that no part of the PPF of a firm that is in some cube after step 3 is removed as a consequence of step 5. Since only one cube is altered by step 5, attention may be focused there. Step 5 can do three different things to the cube, c^ℓ, which is to be divided, depending on the circumstances. In case(a), c^ℓ is removed, and two cubes:

$$c_\alpha^\ell \equiv [(b^{1,\ell}, \ldots, b^{m,\ell}); (w^{1,\ell}, \ldots w^{j-1,\ell}, \frac{b^{j,\ell} + w^{j,\ell}}{2}, w^{j+1,\ell}, \ldots, w^{m,\ell})]$$

and

$$c_\beta^\ell \equiv [(b^{1,\ell}, , \ldots, b^{j-1,\ell}, \frac{b^{j,\ell} + w^{j,\ell}}{2}, b^{j+1,\ell}, \ldots, b^{m,\ell}); (w^{1,\ell}, \ldots, w^{m,\ell})]$$

are added to $\tilde{C}_n^\ell$ to form C_n^ℓ. But it is clear that $C(b^\ell, w^\ell) = C(b_\alpha^\ell, w_\alpha^\ell) \cup C(b_\beta^\ell, w_\beta^\ell)$. Then trivially, since no $y \in C(b^\ell, w^\ell)$ is removed, no $y \in \partial Y^\ell \cap C(b^\ell, w^\ell)$ is removed.

In case(b), it need only be shown that the parts of ∂Y^ℓ inside the cube that is to be divided remain inside the resulting cube. Again, this is immediate since this could only be false if there were some $y^\ell \in \partial Y^\ell \cap C(b^\ell, w^\ell)$, and $y^{j,\ell} < \frac{b^{j,\ell} + w^{j,\ell}}{2}$. But this is a violation of the conditions under which (b) is invoked. Similarly, in case (c), failure of the Lemma implies for some $y^\ell \in \partial Y^\ell \cap C(b^\ell, w^\ell)$, $y^{j,\ell} > \frac{b^{j,\ell} + w^{j,\ell}}{2}$, which violates the conditions of the case.

$\bullet$

Now consider the following definitions:

$$\tilde{C}_n \equiv \sum_{\ell=1}^k \tilde{C}_n^\ell \quad \text{and} \quad C_n \equiv \sum_{\ell=1}^k C_n^\ell.$$

These objects are the subject of the next two corollaries. Global cubes are not actually collected, and do they play any role in the definition of the procedure. we use them briefly here to help prove corollaries 2.1 through 2.3. Corollary 2.1 is just the global analog of Lemma 2.

Corollary 2.1 *For all* $y \in \partial Y(\bar{c}_0)$, *if* $y \in \bigcup_{c \in \tilde{C}_n} C(b, w)$, *then* $y \in \bigcup_{c \in C_n} C(b, w)$.

Proof/

By definition, if:

$$y \in \left\{ \partial Y(\bar{c}_0) \bigcap \left\{ \bigcup_{c \in \tilde{C}_n} C(b, w) \right\} \right\}$$

then,

$$\forall\, \ell = 1, \ldots, k, \exists y^\ell \in \left\{ \partial Y^\ell \bigcap \left\{ \bigcup_{c \in \tilde{C}_n^\ell} C(b, w) \right\} \right\} \text{ s.t. } y = \sum_{\ell=1}^k y^\ell.$$

21

Then by Lemma 2,

$$\forall \, \ell = 1, \ldots, k, \; y^\ell \in \bigcup_{c \in C_n^\ell} \mathcal{C}(b, w),$$

and so by definition of C_n,

$$y \in \bigcup_{c \in C_n} \mathcal{C}(b, w).$$

•

Corollary 2.2 says that no feasible point on the PPF is ever discarded if it is at least as good as the tentative plan.

Corollary 2.2 *For all $y \in \partial Y(\bar{c}_0)$, if $y + \omega \succeq p_n + \omega$, then $y \in \bigcup_{c \in C_n} \mathcal{C}(b, w)$.*

<u>Proof/</u>

This will be shown by induction. Since $\bigcup_{c \in C_0} \mathcal{C}(b, w) = \mathcal{C}(b_0, w_0)$ and $p_0 = w_0 \equiv \sum w_0^\ell$, for iteration 0 the statement reads: $\{y \in \partial Y(\bar{c}_0) \mid y + \omega \succeq w_0 + \omega\} \subset \mathcal{C}(b_0, w_0)$. But this is obviously true since

$$\{y \in \partial Y(\bar{c}_0) \mid y + \omega \succeq w_0 + \omega\} \subset \{y \in \mathcal{C}(b_0, w_0) \mid y + \omega \succeq w_0 + \omega\} = \mathcal{C}(b_0, w_0).$$

Assume the statement is true for n. To show that it is true for $n + 1$ we must show that the containment is preserved as C_n is changed into $\tilde{C}_{n+1}$, and as $\tilde{C}_{n+1}$ is changed into C_{n+1}. To see this for the first transition, take any $y \in \bigcup_{c \in C_n} \mathcal{C}(b, w)$ and suppose that $y \notin \bigcup_{c \in \tilde{C}_{n+1}} \mathcal{C}(b, w)$. Then there is a cube $c \in C_n$ such that $y \in \mathcal{C}(b, w)$ but $c \notin \tilde{C}_{n+1}$. But then, according to step 2, if c is removed from C_n then $b + \omega \prec p_{n+1} + \omega$. Since $y \leq b$ for any $y \in \partial Y(\bar{c}_0)$, it follows by monotonicity of preferences that $y + \omega \prec p_{n+1} + \omega$. To prove that containment is preserved during the second transition, it is sufficient to show that for any $y \in \partial Y(\bar{c}_0)$, if $y \in \bigcup_{c \in \tilde{C}_{n+1}} \mathcal{C}(b, w)$, then $y \in \bigcup_{c \in C_{n+1}} \mathcal{C}(b, w)$. But this is immediate from corollary 2.1.

Corollary 2.3 states that the upper estimate at each iteration is indeed an overestimate of all the interesting parts of the PPF.

Corollary 2.3 *For all* $y \in \partial Y(\bar{c}_0)$ *if* $y + \omega \succeq p_n + \omega$, *then* $y \in UE_n$.

<u>Proof/</u>

Notice that $\bigcup_{c \in C_n} \mathcal{C}(b, w) \subset ch(B_n) \equiv UE_n$. Apply corollary 2.2.

•

Lemma 3 shows that the lower estimate at any iteration n is as advertised.

Lemma 3. *For all iterations* n, $LE_n \subset Y(\bar{c}_0)$.

<u>Proof/</u>

This follows directly from the definition of LE_n as the comprehensive hull of feasible points and the assumption of free disposal.

•

Lemma 4 is a technical fact which will be useful in proving future lemmata.

Lemma 4. *For all iterations* n, $W_n^\ell \subset LE_n^\ell$.

<u>Proof/</u>

Assume that for all n, and for all $c^\ell \in C_n^\ell$, there exists an $x^\ell \in X_n^\ell$ such that $x^\ell \in \mathcal{C}(b^\ell, w^\ell)$. But then for all n, and for all $w^\ell \in W_n^\ell$ there exists an $x^\ell \in X_n^\ell$ such that $x^\ell \geq w^\ell$. Therefore, $W_n^\ell \subset ch(X_n^\ell) \equiv LE_n^\ell$.

Thus it only remains to show by induction that the assumption is true. For $n = 0$, this is obvious since c_0^ℓ is the only element of C_0^ℓ, $w_0^\ell \in X_0^\ell$, and obviously $w_0^\ell \in \mathcal{C}(b_0^\ell, w_0^\ell)$. Now suppose that the assumption is true for iteration n. Consider the following two classes of cubes:

23

1) First consider any cube $c^\ell \in C_{n+1}$ such that $c^\ell \in C_n^\ell$. By the induction hypothesis, there is some $x^\ell \in X_n$ such that $x^\ell \in \mathcal{C}(b^\ell, w^\ell)$. But since $b^\ell \in \tilde{B}_{n+1}^\ell$ and $b^\ell \geq x^\ell$, by step 3 of the algorithm, $x^\ell \in X_{n+1}^\ell$

2) Now consider any cube $c^\ell C_{n+1}$ such that $c^\ell \notin C_n^\ell$. This new cube must have been created through the division some cube $\hat{c}^\ell \in C_n^\ell$ at step 5 of the algorithm. In case (a) of step 5, the firm reports a point $x^\ell \in h(\hat{c})$, and this point is added to X_{n+1}^ℓ. But the point x^ℓ is in both of the two new cubes that result from division since it is in on their common boundary. In cases (b) and (c), by the induction hypothesis, there exists an $x^\ell \in X_n^\ell$ such that $x^\ell \in \hat{c}$. But since $x^\ell \in \partial Y^\ell$, by the hypothesis of the case x must still be in the one cube, c^ℓ that results

●

4. Technical Results on Cube Size

The purpose of this section is to give some technical results. Lemmata 5–8, show that it is possible to find an upper bound on $E_n^{\ell*}$, the length of longest edge of any cube associated with the estimate of the ℓth firms production set at iteration n, and to show that this bound decreases as n goes to infinity in a predictable way. The proofs of the results are given in the appendix.

Recall that the effect of fifth step of the algorithm is that some cube is either divided into two cubes (case (a)), or made into a new cube half as big as the original (cases (b) and (c)). We will call the cube or cubes that result from such a division *resultant cubes*. More generally, we will want to keep track of resultant cubes, cubes resulting from divisions of resultant cubes, and so on. The following

24

notation will be used to indicate the pedigree of these classes of cubes. Consider the set of cubes C_n^ℓ and let some unspecified number of iterations pass. Then the sets $C_{n,r}^\ell, C_{n,r^2}^\ell C_{n,r^3}^\ell \ldots$ will refer to the sets of cubes that are the result of a single division of a cube in C_n, two successive divisions, three successive divisions, etc. We may now state Lemma 5.

Lemma 5. *For any $\ell = 1, \ldots, k$, suppose a cube c_r^ℓ results from the division of a cube c^ℓ through step 5 at iteration n of the algorithm. If $j = ILE(c^\ell)$, then*
$$(b_r^{j,\ell} - w_r^{j,\ell}) = \tfrac{1}{2}(b^j - w^j) \le \tfrac{1}{2}E_n^{\ell*}.$$

Lemma 6. *For any $\ell = 1, \ldots, k$, for any $c_{rm}^\ell \in C_{n,r^m}^\ell$, for all $j = 1, \ldots, m$, $b_{rm}^{j,\ell} - w_{rm}^{j,\ell} \le \tfrac{1}{2}E_n^{\ell*}$.*

The point of Lemma 6 is that if a cube is a result of m divisions of some original cube c^ℓ, one of two things must be true: either each of the m edges of the resultant cube have been divided exactly once, or at least one of the edges have been divided twice (or more). In the former case, all the edges of the resultant cube are exactly half the length of those of c^ℓ. Thus, all the edges are half as long or less than $E_n^{\ell*}$, the longest edge of any cube in C_n^ℓ. In the latter case, if some edge is divided twice, then the longest edge must also have been divided twice since it would be the first to be divided a second time. But then all the edges of the resultant cube are less than or equal to half the length of longest edge of the original cube. But this is less than or equal to $\tfrac{1}{2}E_n^{\ell*}$, which proves the Lemma.

Corollary 6.1 *For any $\ell = 1, \ldots, k$, for any $c^\ell \in C_{n,r^{m'}}^\ell$ for $m' \ge m$, and for all $j = 1, \ldots, m$, $b_{r^{m'}}^{j,\ell} - w_{r^{m'}}^{j,\ell} \le \tfrac{1}{2}E_n^{\ell*}$.*

Corollary 6.1 generalizes Lemma 6 to show that the conclusion holds for cubes that are the result of more than m divisions of an original cube.

Lemma 7. *For any $\ell = 1, \ldots, k$, if at some iteration, n, there are at most Q cubes in the set C_n^ℓ, then after $Q(2^m - 1)$ more iterations, there will be at most $Q2^m$*

cubes in the set $C^{\ell}_{[n+Q(2^m-1)]}$ and

$$E^{\ell*}_{[n+Q(2^m-1)]} \leq \frac{1}{2} E^{\ell*}_n.$$

Lemma 7 extends the argument of corollary 6.1 to say that if at iteration n, there are Q cubes, then after $Q(2^m-1)$ more iterations, each of the original Q cubes must have been divided exactly m times, or at least one of the original cubes must have been divided more than m times. In both cases corollary 6.1 may be applied to conclude that the longest edge of any cube in $C^{\ell}_{n+Q(2^m-1)}$ is at most half the length of the longest edge of any cube in C^{ℓ}_n.

Lemma 8. *For any* $\ell = 1, \ldots, k$, *at iteration* $I(t) = (2^m - 1) \sum_{s=0}^{t} 2^{sm}$, $E^{\ell*}_{I(t)} \leq \frac{1}{2^{t+1}} E^{\ell*}_0$.

Lemma 8 builds on Lemma 7 to consider "blocks" of $Q(2^m - 1)$ iterations in order to calculate how many iterations must pass before $E^{\ell*}_n$ is smaller than $\frac{1}{2^t} E^{\ell*}_0$.

5. Results

At last we come to the results. First we show that the procedure is monotonic and gives feasible tentative plans.

Theorem 1. *The set of tentative production plans,* $\{p_n\}_{n=1}^{\infty}$ *are feasible, and mono- tonically increasing in the preference order.*

<u>Proof/</u>

$p_n \in LE_{n-1} \equiv ch(X_{n-1})$. By monotonicity of preferences, $p_n = x$ for some $x \in X_{n-1}$. But by construction, $x = \sum x^{\ell}$ for some $x^{\ell} \in Y^{\ell} \cap c^{\ell}_0$ for $\ell = 1, \ldots, k$. Thus $\bar{p}_n = \bar{x}$ and so $p^{\ell}_n = x^{\ell} \in Y^{\ell} \cap c^{\ell}_0$ or all $\ell = 1, \ldots, k$. Thus for all n, $p_n \in Y(\bar{c}_0)$.

26

Step 1 of the algorithm stipulates that $p_n \in \{x \in LE_{n-1} \mid x + \omega \succeq y + \omega \ \forall \ y \in LE_{n-1}\}$. But p_{n-1} is also an element of LE_{n-1} since only elements of LE_{n-2} that are strictly inferior to p_{n-1} are removed to form LE_{n-1}. It is immediate from the above that $p_n \succeq p_{n-1}$ for all n. Thus, $\{p_n\}_{n=1}^{\infty}$ is monotonically increasing in the preference order.

•

There are several ways to think about convergence. Theorem 2 shows that the minimal distance between the indifference surface through the tentative plan and the optimal plans converges to zero. In addition, we show that $\frac{E_0^{\bullet}\sqrt{m}}{2^{t+1}}$ is an upper bound on this distance at iteration $n = I(t)$. This may be taken as a measure of an upper bound on speed of convergence in the preference ordering that is independent of the utility representation. We can use this bound as a basis of comparison with other procedures.

Theorem 2. If $n \geq I(t)$, $\nu(L(p_n + \omega), U(X^*(\bar{c}_0) + \omega) \leq \frac{E_0^{\bullet}\sqrt{m}}{2^{t+1}}$

Proof/

Consider any $x^* \in X^*(\bar{c}_0)$. By corollary 2.2, $x^* \in \mathcal{C}(b, w)$ for some $c \in C_n$. Then there exist cubes $c^{\ell} = (b^{\ell}, w^{\ell}) \in C_n^{\ell}$ for $\ell = 1, \ldots, k$ such that $\sum_{\ell=1}^{k} b^{\ell} = b \geq x^*$ and $\sum_{\ell=1}^{k} w^{\ell} = w \leq x^*$. But then there are $x^{*,\ell}$ for $\ell = 1, \ldots, k$ such that $b^{\ell} \geq x^{\ell *} \geq w^{\ell}$ and $\sum_{\ell=1}^{k} x^{\ell *} = x^*$. Since by Lemma 4, $w^{\ell} \in LE_n^{\ell}$, it follows that:

$$\nu(LE_n^{\ell}, x^{\ell *}) \leq \| x^{\ell *} - w^{\ell} \| \leq \| b^{\ell} - w^{\ell} \|$$

But by Lemma 8, $b^{i,\ell} - w^{i,\ell} \leq \frac{E_0^{\ell \bullet}}{2^{t+1}}$ for all $c^{\ell} \in C_n^{\ell}$, $i = 1, \ldots, m$, and $\ell = 1, \ldots, k$. Then using the definition of Euclidean distance:

$$\nu(LE_n^{\ell}, x^{*,\ell}) \leq \sqrt{\sum_{i=1}^{m}(b^{i,\ell} - w^{i,\ell})^2} \leq \sqrt{m\left(\frac{E_0^{\ell *}}{2^{t+1}}\right)^2} = \frac{E_0^{\ell *}\sqrt{m}}{2^{t+1}},$$

27

and so,

$$\nu(LE_n, x^*) \leq \sum_{\ell=1}^{k} \frac{E_0^{\ell*} \sqrt{m}}{2^{t+1}} = \frac{E_0^* \sqrt{m}}{2^{t+1}}.$$

But $x^* \in U(X^*(\bar{x}) + \omega)$. So by Lemma 1, $\nu(LE_n, U(X^*(\bar{c}_0) + \omega)) \leq \frac{E_0^* \sqrt{m}}{2^{t+1}}$. Similarly, by step 1 of the algorithm, and Lemma 3, $LE_n + \omega \subseteq L(p_n + \omega)$. Thus, by Lemma 1, $\nu(L(p_n + \omega), U(X^*(\bar{c}_0) + \omega)) \leq \frac{E_0^* \sqrt{m}}{2^{t+1}}$.

•

In a planning environment, the CPB is usually assumed to know the SWF. It is more traditional, as a consequence, to think of convergence in utility terms. Theorem 3 shows that for any utility representation of social preferences, the utility of the tentative plans converges to the utility of the optimal plans.

Theorem 3. *Given any continuous utility representation, $u : X \to \Re$, of the social preference relation, $\succeq$, for all $x^* \in X^*(\bar{c}_0)$, $\lim_{n \to \infty} u(p_n + \omega) = u(x^* + \omega)$.*

<u>Proof/</u>

Assumptions A1, and A2 are sufficient, according to Debreu(1954), to assure that $\succeq$ can be represented by a continuous utility function. Recall that by Theorem 2, for $n \geq I(t)$, $\nu(L(p_n + \omega), U(X^*(\bar{c}_0) + \omega)) \leq \sum_{\ell=1}^{k} \frac{E_0^{\ell*} \sqrt{m}}{2^{t+1}}$. Thus, as $n \to \infty$, $\nu(L(p_n + \omega), U(x^* + \omega)) \to 0$. Then we can pick two sequences $\{L_n\}$, and $\{U_n\}$ such that for all n, $L_n \in L(p_n + \omega)$, $U_n \in U(p_n + \omega)$, and $\| L_n - U_n \| \to 0$. Since u is continuous, $\lim_{n \to \infty} u(L_n) - u(U_n) = 0$. Then, since $u(L_n) \leq u(p_n + \omega) \leq u(x^* + \omega) \leq u(U_n)$, $\lim_{n \to \infty} u(p_n + \omega) - u(x^* + \omega) = 0$.

•

Theorem 4 gives an upper bound on the speed of convergence in utility terms. Such an estimate is important because it gives the CPB a basis to compare different planning procedures and decide which is best to solve his specific problem.

Recall from real analysis that all real continuous functions on a compact metric space are also uniformly continuous. We will be forced to strengthen this somewhat in order to actually calculate a bound on the speed of convergence. In particular, we will assume that the SWF is *proportionally* uniformly continuous. In economic terms this means essentially that if two indifference curves are close to each other somewhere, then there is a proportionate bound on how far apart they can ever get from one another in the rest of the goods space. This is a bound of the relative steepness and shallowness of the utility hill. This is only a slight strengthening of the existing assumptions on the economy, and as such should be viewed as being purely technical in nature. Given that we already have continuity and strong monotonicity, we are only eliminating the possibility that the utility hill has infliction points by this assumption.

A3) For a given utility representation, u, of $\succeq$, $\exists \, \lambda > 0$ such that $\forall \, x \in \mathcal{C}(b_0, w_0)$, if

$\parallel x - y \parallel \, \leq \, \epsilon$, then $\mid u(x + \omega) - u(y + \omega) \mid \, \leq \lambda \epsilon.$

(Proportional uniform continuity of utility.)

Since we assume that the CPB knows his SWF, it is not significant that he needs to know the λ parameter of proportional uniform continuity to calculate the bound on speed of convergence in utility terms. Note that $\lambda = 1$ if the CPB has transferable utility.

Theorem 4. *For any utility representation, u, of $\succeq$ satisfying A3, any $x^* \in X^*(\bar{c}_0)$, and any $n \geq I(t)$, $u(x^* + \omega) - u(p_n + \omega) \leq \lambda \frac{E_0^* \sqrt{m}}{2^{t+1}}.$*

<u>Proof/</u>

By Theorem 2 for any $n \geq I(t)$, and for all $x^* \in X^*(\bar{c}_0)$, $\nu(L(p_n + \omega), U(x^* + \omega)) \leq \frac{E_0^* \sqrt{m}}{2^{t+1}}$. Thus, for every $n \geq I(t)$, there exist x_n, and y_n such that $x^n + \omega \in U(x^* + \omega)$, $y^n + \omega \in L(p_n + \omega)$ and $\parallel x_n - y_n \parallel \leq \frac{E_0^* \sqrt{m}}{2^{t+1}}$. But then by assumption A3, and the fact the $x_n + \omega \succeq y_n + \omega$, $u(x_n + \omega) - u(y_n + \omega) \leq \lambda \frac{E_0^* \sqrt{m}}{2^{t+1}}$. Therefore, since $x_n + \omega \succeq x^* + \omega$ and $y_n + \omega \preceq p_n + \omega$, the theorem is proved.

29

Finally, Theorem 5 show that the procedure converges not only in utility, but in addition, the tentative plans converge in quantity terms to actual optimal plans.

Theorem 5. *the sequence of tentative plans $\{p_n\}$ converges to the set of optimal plans $X^*(\bar{c}_0)$.*

<u>Proof/</u>

Since $\{p_n\}$ is drawn from the compact set $\mathcal{C}(b_0, w_0)$, we need only show that the limit point of every convergent subsequence is an element of $X^*(\bar{c}_0)$. So take any convergent subsequence $\{p_t\}$ and suppose that $p_t \to p^*$. For any $x^* \in X^*(\bar{c}_0)$, by Theorem 3, $u(p_t + \omega) \to u(x^* + \omega)$. Then by continuity $u(p^* + \omega) = u(x^* + \omega)$. By the definition of p_t, for all t there is a decomposition $\bar{p}_t$, such that for all $\ell = 1, \ldots, k$, $p_t^\ell \in Y^\ell \cap \mathcal{C}(b_0, w_0)$. Since $Y^\ell \cap \mathcal{C}(b_0, w_0)$ is compact by assumption B2, there is a decomposition $\bar{p}^*$, such that for all $\ell = 1, \ldots, k$, $p^{\ell *} \in Y^\ell \cap \mathcal{C}(b_0, w_0)$. Therefore, $p^* \in Y(\bar{c}_0)$, and is also undominated in the preference ordering. We conclude that $p^* \in X^*(\bar{c}_0)$.

Unfortunately, it does not seem to be possible to extend theorem 4 and find a general bound on the speed of convergence in quantity terms. To do so we would have to know much more about the interactions between the preferences and the feasible set. For some subclasses of economies (convex economies for example), it may be possible to find useful characterizations of these interactions. But this will not be attempted in the current paper.

Finally, we turn to the question of stopping rules. Except in very special cases, finite convergence cannot be expected. So in practice, the CPB will have abandon the search at some point and produce the current tentative plan. The CPB must therefore devise a rule to stop the procedure when then the tentative

plan is sufficiently "close" to an optimal plan. If the CPB is satisfied with defining "closeness" in utility terms, stopping rules are very easy to implement. All he need do is subtract the utility of the tentative plan from the utility of the most preferred element of the Upper Estimate, and stop the search when this number falls below a pre-specified threshold. If the CPB insists on making a stopping rule in quantity terms, then things are slightly more complicated. The CPB must find an upper bound on the distance in goods space between the current tentative plan and the set of optimal plans. Recall that the algorithm discards information as it progresses. Then since we know that all $X^*(\bar{c}_0) \in \bigcup_{c \in C_n} \mathcal{C}(b, w)$, one way to find an upper bound is to take the maximum distance between the points in this union. Unfortunately, it will not always be the case that this bound goes to zero as the number of iterations goes to infinity. So a CPB who uses a quantity stopping rule like this can never be sure that he will ever actually stop. Stopping is more likely, however, the closer set of optimal plans are to one another. In particular, if the set of optimal plans is a singleton, stopping is always guaranteed.

6. Conclusions

In conclusion, the new procedure is well defined, monotonic, convergent, and even in a finite number of iterations, if the true production set is a step function.[11] In addition, The message space is simple, and the large class of comprehensive and monotonic economies can be treated. An important feature of the procedure presented in this paper is its ability to discriminate against irrelevant parts of the

---

[11] In Malinvaud's and Weitzman's procedure, finite convergence is obtained when the production set in polyhedral. This is because in this case the production set can be *exactly* approximated by the estimates that the procedure generates. The same thing holds here for step function production sets which can be exactly approximated by Quantity-Quantity type procedures. This is not a very important case, however, and no proof will be offered of this assertion.

goods space and so narrow the area of search. It is unfortunate that there is no obvious way to incorporate this into the estimate of the speed of convergence. The greatest advantage of this procedure is that its design makes it makes it possible to directly calculate an upper bound on the number of iterations it takes for the tentative plans to be within any given ϵ in utility of the optimal plans. Although this is a very loose upper bound, it still can serve as a basis of comparison for speed of convergence with other procedures. This is essential if a CPB is to make an informed decision about which planning procedure to use. In addition, at any given iteration, it is possible to place an upper bound on how far the current tentative plan is from an optimal plan both in utility terms, and in the actual goods space. This allows the CPB to make a reasonable judgment about whether the current plan is good enough, or to continue searching.

Appendix

Lemma 5. For any $\ell = 1, \ldots, k$, suppose a cube c_r^ℓ results from the division of a cube c^ℓ through step 5 at iteration n of the algorithm. If $j = ILE(c^\ell)$, then $(b_r^{j,\ell} - w_r^{j,\ell}) = \frac{1}{2}(b^j - w^j) \leq \frac{1}{2}E_n^{\ell*}$.

Proof/

   Consider any cube c_r^ℓ that results from a division along the jth edge of a cube c^ℓ. The cube c_r^ℓ must have resulted from application of one of cases (a), (b), or (c) in step 5. In case (a), c_r^ℓ can take two forms:

$$\left[(b^{1,\ell}, \ldots, b^{j,\ell}, \ldots, b^{m,\ell}), (w^{1,\ell}, \ldots, \frac{b^{j,\ell} + w^{j,\ell}}{2}, \ldots, w^{m,\ell})\right]$$

or

$$\left[(b^{1,\ell}, \ldots, \frac{b^{j,\ell} + w^{j,\ell}}{2}, \ldots, b^{m,\ell}), (w^{1,\ell}, \ldots, w^{j,\ell}, \ldots, w^{m,\ell})\right].$$

In case (b), only one form is possible:

32

$$\left[(b^{1,\ell},\ldots,b^{j,\ell},\ldots,b^{m,\ell}),(w^{1,\ell},\ldots,\frac{b^{j,\ell}+w^{j,\ell}}{2},\ldots,w^{m,\ell})\right]$$

Likewise, in case (c), c_r must take the form:

$$\left[(b^{1,\ell},\ldots,\frac{b^{j,\ell}+w^{j,\ell}}{2},\ldots,b^{m,\ell}),(w^{1,\ell},\ldots,w^{j,\ell},\ldots,w^{m,\ell})\right].$$

Then since either,

(i) $b_r^{j,\ell}-w_r^{j,\ell}=b^{j,\ell}-\frac{b^{j,\ell}+w^{j,\ell}}{2}=\frac{b^{j,\ell}-w^{j,\ell}}{2}$ or

(ii) $b_r^{j,\ell}-w_r^{j,\ell}=\frac{b^{j,\ell}-w^{j,\ell}}{2}-w^{j,\ell}=\frac{b^{j,\ell}+w^{j,\ell}}{2}$.

But since a cube is always divided along it longest edge, and by definition, no edge of any cube can be longer that $E_n^{\ell*}$ at iteration n, $(b_r^{j,\ell}-w_r^{j,\ell})=\frac{1}{2}(b^{j,\ell}-w^{j,\ell})\le\frac{1}{2}E_n^{\ell*}$.

•

Lemma 6. For all $\ell=1,\ldots,k$, for any $c_{rm}^\ell\in C_{n,rm}^\ell$, for all $j=1,\ldots,m$, $b_{rm}^{j,\ell}-w_{rm}^{j,\ell}\le\frac{1}{2}E_n^{\ell*}$.

Proof/

We start by distinguishing two exhaustive subclasses of cubes in $C_{n,rm}^\ell$.

1. First consider cubes $c_{rm}^\ell\in C_{n,rm}^\ell$ for which no edge has ever been subjected to two separate divisions. That is, cubes that are the result of m divisions of an original cube in C_n^ℓ, and for which each of the m edges has been divided exactly once. By Lemma 5, $(b_{rm}^{j,\ell}-w_{rm}^{j,\ell})=\frac{1}{2}(b^{j,\ell}-w^{j,\ell})$ for all $j=1,\ldots,m$. But by definition, $(b^{j,\ell}-w^{j,\ell})\le E_n^{\ell*}$ for all $j=1,\ldots,m$. Thus, for cubes in this class, $(b_{rm}^{j,\ell}-w_{rm}^{j,\ell})\le\frac{1}{2}E_n^{\ell*}$ for $j=1,\ldots,m$.

2. Now consider cubes $c_{rm}^\ell\in C_{n,rm}^\ell$ for which at least one edge has been subjected to at least two separate divisions. Without loss of generality, suppose that edge j is the longest edge of the original cube c^ℓ, and so is the first to be divided. Then by Lemma 5, $(b_r^{j,\ell}-w_r^{j,\ell})=\frac{1}{2}(b^{j,\ell}-w^{j,\ell})$. But by assumption, $\frac{1}{2}(b^{j,\ell}-w^{j,\ell})\ge\frac{1}{2}(b^{i,\ell}-w^{i,\ell})$ for all $i\ne j$. So if any edge is divided twice through step 5, then the jth edge is also divided twice. But since only the largest edge is ever subject to division, $\frac{1}{2}E_n^{\ell*}\ge\frac{1}{2}(b^{j,\ell}-w^{j,\ell})\ge(b_{rm}^{i,\ell}-w_{rm}^{i,\ell})$ for all $i=1,\ldots,m$. Thus, for cubes in this class as well, $(b_{rm}^{j,\ell}-w_{rm}^{j,\ell})\le\frac{1}{2}E_n^{\ell*}$ for $j=1,\ldots,m$. The Lemma is proven.

•

Corollary 6.1 For all $\ell=1,\ldots,k$, for any $c^\ell\in C_{n,rm'}^\ell$ for $m'\ge m$, and for all $j=1,\ldots,m$, $b_{rm'}^{j,\ell}-w_{rm'}^{j,\ell}\le\frac{1}{2}E_n^{\ell*}$.

Proof/

33

By Lemma 6, for any $c_{rm}^\ell \in C_{n,rm}^\ell$, for all $j = 1,\ldots,m$, $b_{rm}^{j,\ell} - w_{rm}^{j,\ell} \leq \frac{1}{2}E_n^{\ell*}$. Since it is impossible for any edge of any cube to be increased as a result of a division, $b_{rm'}^{j,\ell} - w_{rm'}^{j,\ell} \leq \frac{1}{2}E_n^{\ell*}$ for $m' \geq m$.

$\bullet$

Lemma 7. *For all $\ell = 1,\ldots,k$, if at some iteration n there are at most Q cubes in the set C_n^ℓ, then after $Q(2^m - 1)$ more iterations, there will be at most $Q2^m$ cubes in the set $C_{[n+Q(2^m-1)]}^\ell$ and*

$$E_{[n+Q(2^m-1)]}^{\ell*} \leq \frac{1}{2}E_n^{\ell*}.$$

Proof/

The first part is easy to show. Each iteration can add at most one cube. This is because the step 3 eliminates cubes, if it does anything at all, and (a) of step 5 adds one additional cube while (b) and (c) leave the number of cubes unchanged. Thus, an upper bound on the number of cubes at the end of iteration $n + Q(2^m - 1)$ when there were at most Q cubes at iteration n is $Q + Q(2^m - 1) = Q2^m$.

To see the second part, suppose initially that each of the cubes in C_n^ℓ is divided once before any of the cubes is redivided. Then after Q iterations, $C_n^\ell \cap C_{[n+Q]}^\ell = \emptyset$. Also, $C_{[m+Q]}^\ell$ will consist of at most $2Q$ cubes. Now let each cube in C_{n+Q}^ℓ be divided once before any is divided a second time. Since $C_{[n+Q]}^\ell$ contains at most $2Q$ cubes, this will take at most $2Q$ more iterations. At the end of these iterations $C_{[n+Q+2Q]}^\ell$ will contain at most $Q2^2$ cubes, and by construction, $C_n \cap C_{[n+Q+2Q]} = \emptyset$ and $C_{n,r}^\ell \cap C_{[n+Q+2Q]}^\ell = \emptyset$. Suppose that this process continues, and each cube in the set C_{n,r^x}^ℓ is divided before any cube in $C_{n,r^{x+1}}^\ell$ is. Then by the end of $n + Q + 2Q + Q2^2 + Q2^3 + \ldots + Q2^{m-1}$ iterations, there are at most $Q2^m$ cubes and

$$C_{n,r^x}^\ell \cap C_{[n+Q+2Q+Q2^2+Q2^3+,\ldots,+Q2^{m-1}]}^\ell = \emptyset$$

for all $x < m$. Note that

$$Q(1 + 2 + 2^2 +,\ldots,+2^{m-1}) = Q(2^m - 1).$$

We conclude that all elements of $C_{[n+Q(2^m-1)]}^\ell$ are elements of some $C_{n,rm'}^\ell$ where $m' \geq m$. But by corollary 6.1, for any $c^\ell \in C_{n,rm'}^\ell$ for all $m' \geq m$, and for all $j = 1,\ldots,m$, $b_{rm'}^{j,\ell} - w_{rm'}^{j,\ell} \leq \frac{1}{2}E_n^{\ell*}$. Therefore:

$$E_{[n+Q(2^m-1)]}^{\ell*} \leq \frac{1}{2}E_n^{\ell*}.$$

34

Now suppose that things do not develop so neatly and some cubes in C^ℓ_{n,r^x+1} are divided before some in C^ℓ_{n,r^x}. Then assume some cubes have been divided less than m times by the end of iteration $n + Q(2^m - 1)$ (or else corollary 6.1 can be applied directly as in the first case). Then since one cube must be divided at each iteration, if some cube is divided less than m times, some other cube must have been divided more than m times by the end of iteration $n + Q(2^m - 1)$. Thus at some stage, (say iteration n' where $n \le n' \le n + Q(2^m - 1)$), some $\tilde{c}^\ell \in C^\ell_{n,rm}$ is divided. But this can only happen if for all $\hat{c}^\ell \in C^\ell_{n'}$, and all $i = 1, \ldots, m$, $\hat{b}^{i,\ell} - \hat{w}^{i,\ell} \le \tilde{b}^{j,\ell} - \tilde{w}^{j,\ell}$, where $j = ILE(\tilde{c}^\ell)$. Since further divisions of cubes after iteration n' will not increase the length of any edge of any cube, and by Corollary 6.1 $\tilde{b}^{j,\ell} - \tilde{w}^{j,\ell} \le \frac{1}{2} E^{\ell*}_n$, we conclude:

$$E^{\ell*}_{[n+Q(2^m-1)]} \le \frac{1}{2} E^{\ell*}_n.$$

•

Lemma 8. *For all $\ell = 1, \ldots, k$, at iteration $I(t) = (2^m - 1)\sum_{s=0}^{t} 2^{sm}$, $E^{\ell*}_{I(t)} \le \frac{1}{2^{t+1}} E^{\ell*}_0$.*

Proof/

This is shown by induction. For $t = 0$, $I(t) = 2^m - 1$, so it must be shown that $E^{\ell*}_{[2^m-1]} \le \frac{1}{2} E^{\ell*}_0$. At iteration $n = 0$, there are $Q = 1$ cubes in C^ℓ_0, so by Lemma 7, after $1 \times (2^m - 1)$ additional iterations,

$$E^{\ell*}_{[0+(2^m-1)]} = E^{\ell*}_{[(2^m-1)]} \le \frac{1}{2} E^{\ell*}_0.$$

Now assume that the statement is true for t. Then $E^{\ell*}_{I(t)} \le \frac{1}{2^{t+1}} E^{\ell*}_0$. Since each iteration adds at most one cube, and there exists only one cube at iteration 0, after $I(t)$ iterations, there are at most

$$I(t) + 1 = (2^m - 1)\left(\sum_{s=0}^{t} 2^{sm}\right) + 1 = \left(\sum_{s=1}^{t+1} 2^{sm} - \sum_{s=0}^{t} 2^{sm}\right) + 1 = 2^{m(t+1)}$$

cubes in $C^\ell_{I(t)}$. Then by Lemma 7, after at most $2^{m(t+1)}(2^m - 1)$ more iterations,

$$E^{\ell*}_{[I(t)+2^{m(t+1)}(2^m-1)]} \le \frac{1}{2} E^{\ell*}_{I(t)} \le \frac{1}{2} \times \frac{1}{2^{t+1}} E^{\ell*}_0 \le \frac{1}{2^{t+2}} E^{\ell*}_0$$

But

$$I(t) + 2^{m(t+1)}(2^m - 1) = (2^m - 1)\left(\sum_{s=0}^{t} 2^{sm}\right) + (2^m - 1)2^{m(t+1)} =$$

35

$$(2^m - 1) \left(\sum_{s=0}^{t+1} 2^{sm} \right) = I(t+1).$$

So at I(t+1), $E_{I(t+1)}^{\ell*} \leq \frac{1}{2^{t+2}} E_0^{\ell*}$.

•

References

Bennett, J. (1985): "Planning Under Market Socialism when Iteration is Incomplete," *Journal of Comparative Economics*, **9**, 252-266.

Chander, P. and T. R. Kundu (1991): "Comparing Iterative Planning Procedures," *Journal of Comparative Economics*, **15**, 491-99.

Chander, P. (1978): "On a Planning Process Due to Taylor," *Econometrica*, **46**, 761-777.

Clark, S. (1984): "Informational and Performance Properties of a Class of Iterative Planning Procedures," *Review of Economic Studies*, 615-631.

Clark, S. (1991): "Comparing Iterative Planning Procedures: A Reply to Chander and Kundu," *Journal of Comparative Economics*, **15**, 500-502.

Crémer, J. (1977): "Quantity-quantity algorithm for planning under increasing returns to scale.," *Econometrica*, **45**, 1339-1348.

Crémer, J. (1983): "The Discrete Heal Algorithm with Intermediate Goods.," *Review of Economic Studies*, 383-391.

Debreu, G. (1954): "Representation of a Preference Ordering by a Numerical Function," in *Decision Processes*, by R. M. Thrall, C. H. Coombs, and R. L. Davis, eds. 159-65, Wiley.

Heal, G. (1973): *The Theory of Economic Planning.* North-Holland.

Heal, G. (1986): "Planning," in *Handbook of Mathematical Economics*, by K. Arrow and M. Intrilligator, eds. Vol. 3, 1483-1510, North-Holland.

Henry, C. and A. Zylberberg (1978): "Planning Algorithms to Deal with Increasing Returns," *Review of Economic Studies*, 67-78.

Malinvaud E. (1967): "Decentralized Procedures for Planning," in *Activity Analysis in the Theory of Growth and Planning*, by E. Malinvaud and M. Bacharach, eds. St.Martins's Press.

Starrett, D. (1972): "Fundamental Non-convexities in the Theory of Externalities," *Journal of Economic Theory,* 4, 180-99.

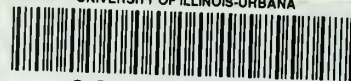**Weitzman, M.** (1970): "Iterative Multilevel Planning With Production Targets," *Econometrica,* **38**, 50-65.